

On the roots of a deformed algebraic system

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Abstract

A problem concerning the shift of roots of a system of algebraic equations is investigated. Its conservation and decomposition of a multiple root into simple roots are discussed.

1 Introduction

The central subject of the present paper is an investigation on the shift of roots of the system of algebraic equations. Our central Theorem 3.1 states that the number of real roots of a system located in a compact set does not change after a sufficiently small perturbation of the system.

As a matter of fact, this kind of fact has been well known to those who study the deformation of the singularities of differentiable mappings. It is, however, a non trivial question how small this perturbation shall be so that the number of roots in a given compact set remains unchanged. All the ever existing theorems (see [2] §12.6) do not precise the size of the compact set and the perturbation of the system under question. They state simply that for a compact set and perturbation, both of them small enough, the invariance of the number of roots holds. This situation can be explained by the fact that they simply treat the notion of local algebra, and consequently they are valid only in the germ sense. We try to give here an estimate on the size of admissible perturbation of the system for a fixed compact set.

Furthermore, we give a result about the decomposition of multiple roots into simple roots. In particular, our Theorem 4.2 assures us the existence of a deformed system of the original system that possesses only simple roots.

The paper is organized as follows. In §2 preliminary results are presented. In §3 we state that a slightly deformed system has the same number of zeros as the original system in taking the multiplicities into account. In §4 we give a result about the decomposition of multiple roots into simple roots. The paper ends in §5 with some concluding remarks.

2 Preliminary results

Let us consider the following system of algebraic equations with real coefficients $a_\alpha^{(i)}$:

$$\left\{ \begin{array}{l} f_1(x_1, x_2, \dots, x_n) = 0, \\ f_2(x_1, x_2, \dots, x_n) = 0, \\ \vdots \\ f_n(x_1, x_2, \dots, x_n) = 0, \end{array} \right. \quad (1)$$

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with

$$\begin{aligned}
f_i(x_1, x_2, \dots, x_n) &= \sum_{|\alpha|=m_i} a_\alpha^{(i)} x^\alpha + a_0^{(i)}, \\
x^\alpha &= x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}, \quad a_0^{(i)} \in \mathbb{R}, \\
|\alpha| &= \alpha_1 + \alpha_2 + \dots + \alpha_n,
\end{aligned} \tag{2}$$

where the degrees of polynomials are ordered as follows:

$$m_1 \leq m_2 \leq \dots \leq m_n.$$

Let us consider the situation where the gradient ideal $\left\langle \frac{\partial}{\partial x_1} f_\ell, \dots, \frac{\partial}{\partial x_n} f_\ell \right\rangle$ over $\mathbb{R}[x]$ contains certain power of maximal ideal \mathfrak{m}^k . That is to say,

$$\left\langle \frac{\partial f_\ell(x)}{\partial x_1}, \frac{\partial f_\ell(x)}{\partial x_2}, \dots, \frac{\partial f_\ell(x)}{\partial x_n} \right\rangle \supset \langle x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \rangle_{\alpha_1 + \dots + \alpha_n \geq k}. \tag{3}$$

Let us note by $M_\alpha(x) = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, $\alpha_1 + \dots + \alpha_n = k$. It is possible to consider the set of such monomials as a basis of \mathfrak{m}^k . The dimension $\mu_n(k)$ of the basis of the ideal \mathfrak{m}^k can be calculated by the following recurrent relation:

$$\begin{aligned}
\mu_n(k) &= \sum_{i=0}^k \mu_{n-1}(k-i), \\
\mu_2(k) &= k+1, \\
\mu_3(k) &= \frac{(k+2)(k+1)}{2}, \\
&\vdots
\end{aligned}$$

Evidently, $\mu_n(k)$ is the number of the entire lattice points on an $(n-1)$ -dimensional face of the n -simplex:

$$\mu_n(k) = \# \{ (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n : \alpha_1 + \dots + \alpha_n = k \}.$$

We recall the notation that:

$$\text{supp } \varphi(x) = \{ \alpha \in \mathbb{Z}^n; \varphi_\alpha \neq 0 \},$$

for a Laurent polynomial $\varphi(x) = \sum_{\alpha \in \mathbb{Z}^n} \varphi_\alpha x^\alpha$. We call a germ $\varphi(x)$ convenient at zero when the Newton diagram of it at zero contains non-compact part of all coordinate axes (cf. [2]). In other words $\varphi(x)$ is convenient at zero if it admits the representation,

$$\varphi(x) = \sum_{i=1}^n x_i^{\beta_i} + R(x),$$

for $\beta_i \geq 1$ and a certain polynomial $R(x)$. It is easy to see that if $f_\ell(x)$ has a convenient germ at zero then there exists $k \geq 1$ such that the condition (3) is satisfied.

Suppose that a polynomial vector:

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \vdots \\ \varphi_n(x) \end{pmatrix} = F \cdot \begin{pmatrix} \varphi(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where F is an invertible constant matrix and $\varphi(x) \in \mathfrak{m}^{k+1}$ and $\deg \varphi(x) = k' \geq k+1$, i.e. $\text{supp } \varphi(x) \subseteq \{\alpha \in \mathbb{Z}_{\geq 0}^n; k+1 \leq |\alpha| \leq k'\}$. The question we pose concerns the behavior of roots of a system:

$$\begin{cases} f_1(x) + t\varphi_1(x) = 0, \\ f_2(x) + t\varphi_2(x) = 0, \\ \vdots \\ f_n(x) + t\varphi_n(x) = 0, \end{cases} \quad (4)$$

with $t \in [0, 1] \subset \mathbb{R}$ as a parameter.

To formulate further statements in a proper way, we introduce notations:

$$f_i(x) = a_1^{(i)} x^{\vec{v}_1^{(i)}} + a_2^{(i)} x^{\vec{v}_2^{(i)}} + \cdots + a_{\lambda_i}^{(i)} x^{\vec{v}_{\lambda_i}^{(i)}},$$

where the vectors $\vec{v}_j^{(i)} = (v_{j,1}^{(i)}, v_{j,2}^{(i)}, \dots, v_{j,n}^{(i)})$, $1 \leq j \leq \lambda_i$, satisfy:

$$\langle (1, \dots, 1), \vec{v}_1^{(i)} \rangle = \langle (1, \dots, 1), \vec{v}_2^{(i)} \rangle = \cdots = \langle (1, \dots, 1), \vec{v}_{\lambda_i}^{(i)} \rangle = m_i.$$

In general, it is not easy to formulate a sufficient condition on $f_\ell(x)$ so that the condition (3) holds. We propose here a simple necessary condition for that.

Propositin 2.1 *The following isomorphism (5) is necessary so that the condition (3) holds,*

$$\begin{aligned} \text{lattice spanned over } \mathbb{Z} \text{ of } \left\{ \vec{v}_1^{(\ell)} - \vec{v}_2^{(\ell)}, \dots, \vec{v}_1^{(\ell)} - \vec{v}_{\lambda_\ell}^{(\ell)} \right\} &\cong \\ \left\{ \vec{\alpha} \in \mathbb{Z}^n; \langle (1, \dots, 1), \vec{\alpha} \rangle = 0 \right\}. \end{aligned} \quad (5)$$

Proof If the condition (5) does not hold, it is evidently impossible to create all monomials x^α with $|\alpha| = k$ as a linear combination of $\frac{\partial}{\partial x_i} f_\ell(x)$'s. **Q.E.D.**

Remark 2.1 *This isomorphism can be realized by shifting each lattice point \vec{v} of the right hand side of Relation (5) towards the lattice point $\vec{v} + (m_i, 0, \dots, 0)$.*

We give now an example for which condition (3) does not hold.

Example 2.1 *Let us consider the following system:*

$$\begin{cases} f_1(x_1, x_2) = a_1^{(1)} x_1^6 + a_2^{(1)} x_1^3 x_2^3 + a_3^{(1)} x_2^6 = 0, \\ f_2(x_1, x_2) = a_1^{(2)} x_1^{12} + a_2^{(2)} x_1^6 x_2^6 + a_3^{(2)} x_2^{12} = 0, \end{cases} \quad (6)$$

where:

$$(a_2^{(i)})^2 - 4a_1^{(i)} a_3^{(i)} \neq 0, \quad i = 1, 2.$$

For these polynomials the lattice define on the left hand side of Relation (5) is isomorphic to

$$\left\{ \vec{\alpha} \in \mathbb{Z}^2; \langle (1, 1), \vec{\alpha} \rangle = 0; \frac{\vec{\alpha}}{3} \in \mathbb{Z}^2 \right\}.$$

Thus, in this case, the condition (3) does not hold.

From now on, we use the notation $f_\ell(x, a)$ instead of $f_\ell(x)$ if we want to emphasize its dependence on the coefficients $a = (a_1^\ell, \dots, a_{\lambda_\ell}^\ell)$. For a set of polynomials $A_1^r(x, a), A_2^r(x, a), \dots, A_{\nu_r}^r(x, a) \in \mathbb{R}[x, a]$ homogeneous in variables a we consider the following linear combinations:

$$A_j^{r+1}(x, a) = \sum_{i=1}^{\nu_0} \gamma_i^{(j)}(a) x^{\vec{\beta}_i^{(j)}} A_i^r(x, a), \quad j = 1, 2, \dots, \nu_{r+1},$$

where $\vec{\beta}_i^{(j)} \in \mathbb{Z}_{\geq 0}^n$ and $\gamma_i^{(j)}(a)$ are linear polynomials in variables a .

Propositin 2.2 *Let as consider the chain of polynomials sets:*

$$\left\{ \Lambda_1^r(x, a), \Lambda_2^r(x, a), \dots, \Lambda_{\nu_r}^r(x, a) \right\}, \quad r = 0, 1, 2, \dots$$

as above with

$$\Lambda_1^0(x, a) = \frac{\partial}{\partial x_1} f_\ell(x, a), \dots, \Lambda_n^0(x, a) = \frac{\partial}{\partial x_n} f_\ell(x, a).$$

Suppose that for certain $r = L$, some of $\Lambda_*^L(x, a)$'s coincides with $\lambda^{(s)}(a)M_s(x)$. That is to say there exists $\bar{h}_1^{(s)}(x, a), \dots, \bar{h}_n^{(s)}(x, a) \in \mathbb{R}[x, a]$ such that

$$\lambda^{(s)}(a)M_s(x) = \sum_{i=1}^n \bar{h}_i^{(s)}(x, a) \frac{\partial}{\partial x_i} f_\ell(x, a).$$

Then $\deg_a \lambda^{(s)}(a) = L$ and $\deg_a \bar{h}_i^{(s)}(x, a) = L - 1$.

Proof After the definition of the recursive process to create $\Lambda_j^{r+1}(x, a)$ from $\Lambda_j^r(x, a)$ it is clear that $\Lambda_j^r(x, a)$ is a homogeneous polynomial of degree r in a . The statement is the direct consequence of this fact.

Q.E.D.

Example 2.2 *We consider the following example*

$$f_\ell(x_1, x_2, a) = f(x, a) = a_1 x_1^5 + a_2 x_1^2 x_2^3 + a_3 x_2^5 = 0,$$

with

$$\begin{aligned} \frac{\partial}{\partial x_1} f(x, a) &= 5a_1 x_1^4 + 2a_2 x_1 x_2^3, \\ \frac{\partial}{\partial x_2} f(x, a) &= 3a_2 x_1^2 x_2^2 + 5a_3 x_2^4. \end{aligned}$$

Then we have the following chain of polynomials to get $x^{\bar{\alpha}_2} = x_1^2 x_2^8$ as a linear combination of $\theta_1 = \frac{\partial}{\partial x_1} f(x, a)$ and $\theta_2 = \frac{\partial}{\partial x_2} f(x, a)$:

$$\begin{aligned} \Lambda_1^1(x, a) &:= (2a_2 x_1 x_2^5 - 5a_1 x_1^4 x_2^2) \theta_1, \\ \Lambda_2^1(x, a) &:= (3a_2 x_1^6 - 5a_3 x_1^4 x_2^2) \theta_2, \\ \Lambda_3^1(x, a) &:= 5a_3 x_1^2 x_2^4 \theta_2, \\ \Lambda_1^2(x, a) &:= 3a_2 \Lambda_1^1(x, a), \\ \Lambda_2^2(x, a) &:= 3a_2 \Lambda_2^1(x, a) + 5a_3 \Lambda_3^1(x, a), \\ \Lambda_1^3(x, a) &:= 3a_2 \Lambda_1^2(x, a), \\ \Lambda_2^3(x, a) &:= 5a_1 \Lambda_2^2(x, a), \\ \Lambda_1^4(x, a) &:= 3a_2 \Lambda_1^3(x, a) + 5a_1 \Lambda_2^3(x, a) = (2^2 3^3 a_2^5 + 5^5 a_1^2 a_2^3) x_1^2 x_2^8. \end{aligned}$$

Thus we have:

$$\begin{aligned} \lambda^{(2)}(a) &= 2^2 3^3 a_2^5 + 5^5 a_1^2 a_2^3, \\ \bar{h}_1^{(2)}(x, a) &= (3a_2)^3 (2a_2 x_1 x_2^5 - 5a_1 x_1^4 x_2^2), \\ \bar{h}_2^{(2)}(x, a) &= (5a_1)^2 ((3a_2)^2 x_1^6 - 15a_2 a_3 x_1^4 x_2^2 - (5a_3)^2 x_1^2 x_2^4). \end{aligned}$$

For the case of $x^{\bar{\alpha}^5} = x_1^5 x_2^5$, we have:

$$\begin{aligned}\lambda^{(5)}(a) &= 5a_1(2^2 3^3 a_2^5 + 5^5 a_1^2 a_2^3), \\ \bar{h}_1^{(5)}(x, a) &= 5a_1^2 a_2^3 x_1 x_2^5 + 2 \cdot 3^3 \cdot 5a_1 a_2^4 x_1^4 x_2^2, \\ \bar{h}_2^{(5)}(x, a) &= -2 \cdot 5^2 a_1^2 a_2((3a_2)^2 x_1^6 - 15a_2 a_3 x_1^4 x_2^2 - (5a_3)^2 x_1^2 x_2^4).\end{aligned}$$

For the case of $x^{\bar{\alpha}^{10}} = x_2^{10}$, we have:

$$\begin{aligned}\lambda^{(10)}(a) &= 5a_3(2^2 3^3 a_2^5 + 5^5 a_1^2 a_2^3), \\ \bar{h}_1^{(10)}(x, a) &= (2^2 3^3 a_2^5 + 5^5 a_1^2 a_2^3) x_2^6 - \\ &\quad - 3a_2(5a_1)^2((3a_2)^2 x_1^6 - 15a_2 a_3 x_1^4 x_2^2 - (5a_3)^2 x_1^2 x_2^4), \\ \bar{h}_2^{(10)}(x, a) &= -(3a_2)^4(2a_2 x_1 x_2^5 - 5a_1 x_1^4 x_2^2).\end{aligned}$$

3 The number of roots of a deformed system

In this section we state that a slightly deformed system has the same number of zeros as the original system in taking the multiplicities into account. We recall here that the index $\ell \in [1, n]$ has been fixed so that $f_\ell(x)$ satisfies the condition (3).

Definition 3.1 We introduce the norm:

$$\|\varphi\| = \sum_{\alpha \in \text{supp } \varphi_\ell(x)} |\alpha| |\varphi_{\ell, \alpha}|,$$

where $\varphi_\ell(x) = \sum_{\alpha \in \text{supp } \varphi_\ell(x)} \varphi_{\ell, \alpha} x^\alpha$. We name the following value by $C(a)$:

$$C(a) = \max_{1 \leq s \leq \mu} \left(\sum_{1 \leq j \leq n} \sum_{|\vec{\beta}| \leq k' - k - 1} \max_{x \in \mathbf{K}} |h_{j, \ell}^{(s)}(x, a) x^{\vec{\beta}}| \right),$$

for some compact set \mathbf{K} and $h_{j, \ell}^{(s)}(x, a) = \frac{\bar{h}_{j, \ell}^{(s)}(x, a)}{\lambda^{(s)}(a)}$ after the notation of Proposition 2.2. \square

Remark 3.1 In general we can not give any reasonable estimate on $C(a)$. In the above Example 2.2 $h_{j, \ell}^{(s)}(x, a)$ contains coefficients of the form:

$$\frac{\text{polynomial of degree 5 in } (a_1, a_2, a_3)}{a_3(2^2 3^3 a_2^5 + 5^5 a_1^2 a_2^3)}.$$

This value can be as large as possible if the denominator is very near to zero. The coefficients of $h_{s, i}^j(x, a)$ contain rational functions in the variable a , with denominators $\lambda^{(s)}(a)$ introduced in Proposition 2.2. \square

Before formulating our main theorem, we recall a simple lemma of linear algebra.

Lemma 3.1 Let us consider $\mu \times \mu$ matrix $A = (a_{ij}) \in \text{End}(\mathbb{R}^\mu)$. If $|a_{ij}| < \frac{1}{\mu^2}$, then $(\text{id}_\mu + A)$ is invertible.

Proof By straightforward calculation of the determinant of $(\text{id}_\mu + A)$ we have:

$$\det(\text{id}_\mu + A) = 1 + a_{11} + a_{22} + \cdots + a_{\mu\mu} + R(a),$$

where $R(a)$ is a polynomial containing $(\mu^2 - \mu - 1)$ terms of monomials in (a_{ij}) whose degrees are higher than or equal to two and less than or equal to μ . Evidently, under the condition $|a_{ij}| < \frac{1}{\mu^2}$ we obtain that $\det(\text{id}_\mu + A) \neq 0$. **Q.E.D.**

Theorem 3.1 *The number of simple roots of the system (4) inside of a compact set \mathbf{K} coincides with that of the system (1) if t satisfies the following inequality:*

$$t < \frac{1}{\|\varphi\| C(a) \mu^2}. \quad (7)$$

Proof Our strategy consists in the construction of a homotopy that connects the simple roots of system (1) and those of (4).

Suppose that we succeed in constructing a homotopy $x(\tau)$, $0 \leq \tau \leq t$ with $x(0) = x$ such that

$$f_s(x(\tau)) + \tau \varphi_s(x(\tau)) = f_s(x), \quad 1 \leq s \leq n,$$

then the vector field along it satisfies the following equality:

$$\frac{d}{d\tau} \begin{pmatrix} f_1(x(\tau)) \\ f_2(x(\tau)) \\ \vdots \\ f_n(x(\tau)) \end{pmatrix} = \sum_{i=1}^n \dot{x}_i(\tau) \frac{\partial}{\partial x_i} \begin{pmatrix} f_1(x(\tau)) \\ f_2(x(\tau)) \\ \vdots \\ f_n(x(\tau)) \end{pmatrix}.$$

In applying this relation to system (4), we get,

$$\sum_{i=1}^n \dot{x}_i(\tau) \frac{\partial}{\partial x_i} \left\{ \begin{pmatrix} f_1(x(\tau)) \\ f_2(x(\tau)) \\ \vdots \\ f_n(x(\tau)) \end{pmatrix} + \tau F \begin{pmatrix} \varphi(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\} + F \begin{pmatrix} \varphi(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Further, we shall realize a smooth homotopy

$$\varphi_\ell(x) = \left(\sum_{i=1}^n v_i(x, \tau) \frac{\partial}{\partial x_i} \right) (f_\ell(x) + \tau \varphi_\ell(x)).$$

We remember that we denoted the basis of \mathfrak{m}^k by $M_{\alpha_i}(x)$, $1 \leq i \leq \mu = \mu_n(k)$. The condition (3) entails the following relation:

$$\begin{pmatrix} M_{\alpha_1}(x) \\ M_{\alpha_2}(x) \\ \vdots \\ M_{\alpha_\mu}(x) \end{pmatrix} = H^{(1)} \begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \frac{\partial f_2}{\partial x_1} \\ \vdots \\ \frac{\partial f_n}{\partial x_1} \end{pmatrix} + H^{(2)} \begin{pmatrix} \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_2} \\ \vdots \\ \frac{\partial f_n}{\partial x_2} \end{pmatrix} + \dots + H^{(n)} \begin{pmatrix} \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_n} \\ \vdots \\ \frac{\partial f_n}{\partial x_n} \end{pmatrix}, \quad (8)$$

for some polynomial entry rank-1 ($\mu \times n$) matrices $H^{(1)}, H^{(2)}, \dots, H^{(n)}$ of the form:

$$H^{(i)} = \mu \overbrace{\left\{ \begin{pmatrix} 0 & \dots & h_{i,\ell}^{(1)} & 0 & \dots & 0 \\ 0 & \dots & h_{i,\ell}^{(2)} & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & h_{i,\ell}^{(\mu)} & 0 & \dots & 0 \end{pmatrix} \right\}}^n,$$

where $h_{j,\ell}^{(s)}(x, a) = \frac{\bar{h}_j^{(s)}(x, a)}{\lambda^{(s)}(a)}$ after the notation of Proposition 2.2 concentrated at the ℓ -th column of the matrix $H^{(i)}$. One rewrites the relation (8) as follows:

$$\begin{pmatrix} M_{\alpha_1}(x) \\ M_{\alpha_2}(x) \\ \vdots \\ M_{\alpha_\mu}(x) \end{pmatrix} = \left(\sum_{i=1}^n H^{(i)} \frac{\partial}{\partial x_i} \right) \begin{pmatrix} f_1 + \tau \varphi_1 \\ f_2 + \tau \varphi_2 \\ \vdots \\ f_n + \tau \varphi_n \end{pmatrix} -$$

$$-\tau \left(\sum_{i=1}^n H^{(i)} F \frac{\partial}{\partial x_i} \right) \begin{pmatrix} \varphi(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (9)$$

As we supposed that $\varphi(x) \in \mathfrak{m}^{k+1}$, it is easy to see that:

$$\left(\sum_{i=1}^n H^{(i)} F \frac{\partial}{\partial x_i} \right) \begin{pmatrix} \varphi(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = A(x) \begin{pmatrix} M_{\alpha_1}(x) \\ M_{\alpha_2}(x) \\ \vdots \\ M_{\alpha_\mu}(x) \end{pmatrix}, \quad (10)$$

with certain polynomial $(\mu \times \mu)$ matrix $A(x)$, where:

$$A(x) = \begin{pmatrix} g_1^{(1)}(x) & \cdots & g_1^{(\mu)}(x) \\ \vdots & \ddots & \vdots \\ g_\mu^{(1)}(x) & \cdots & g_\mu^{(\mu)}(x) \end{pmatrix}.$$

By recalling (9), we obtain an equation as follows:

$$(id_\mu + \tau A(x)) \begin{pmatrix} M_{\alpha_1}(x) \\ M_{\alpha_2}(x) \\ \vdots \\ M_{\alpha_\mu}(x) \end{pmatrix} = \left(\sum_{i=1}^n H^{(i)} \frac{\partial}{\partial x_i} \right) \begin{pmatrix} f_1 + \tau \varphi_1 \\ f_2 + \tau \varphi_2 \\ \vdots \\ f_n + \tau \varphi_n \end{pmatrix}.$$

Supposing that τ is very small, get the inverse to:

$$(id_\mu + \tau A(x)), \quad (11)$$

in the domain $\{x; \det(id_\mu + \tau A(x)) \neq 0\}$. The inequality (7) ensures the invertibility of the matrix (11). To show this, in view of Lemma 3.1, it is enough to verify that for such a value of τ we have:

$$\tau \left(\max_{1 \leq i, s \leq \mu} \max_{x \in \mathbf{K}} |g_s^{(i)}(x)| \right) < \frac{1}{\mu^2}. \quad (12)$$

In other words, it is enough to prove that:

$$\max_{1 \leq i, s \leq \mu} \max_{x \in \mathbf{K}} |g_s^{(i)}(x)| < \|\varphi\| C(a). \quad (13)$$

We remember that $\text{supp } g_s^{(i)}(x) \subset \{\alpha \in \mathbb{Z}^n; k' - m_n \leq |\alpha| \leq k' - m_i\}$. This is a direct consequence of (10). As we have $\text{supp } \varphi_\ell(x) \subset \{\alpha \in \mathbb{Z}^n; k + 1 \leq |\alpha| \leq k'\}$, we can find for every $1 \leq \lambda \leq n$ a series of polynomials $\xi_{\ell, \lambda}^{(1)}(x), \dots, \xi_{\ell, \lambda}^{(\mu)}(x)$ such that:

$$\frac{\partial}{\partial x_\lambda} \varphi_\ell(x) = \sum_{c=1}^{\mu} \xi_{\ell, \lambda}^{(c)}(x) M_c(x). \quad (14)$$

In terms of these polynomials:

$$g_s^{(c)}(x) = \sum_{\lambda=1}^n h_{\lambda, \ell}^{(s)}(x) \xi_{\ell, \lambda}^{(c)}(x),$$

and

$$\text{supp } \xi_{\ell, \lambda}^{(c)}(x) \subset \{\alpha \in \mathbb{Z}^n; 0 \leq |\alpha| \leq k' - k - 1\},$$

the absolute value of each coefficient of $\xi_{\ell,\lambda}^{(c)}(x)$ can be estimated by $\|\varphi\|$ after (14) above. We replace $\xi_{\ell,\lambda}^{(c)}(x)$ by $\|\varphi\| \times \left(\sum_{\vec{\beta} \in \text{supp } \xi_{\ell,\lambda}^{(c)}(x)} x^{\vec{\beta}} \right)$ and we get the inequality:

$$\max_{1 \leq i, s \leq \mu} \max_{x \in \mathbf{K}} |g_s^{(i)}(x)| < \left(\sum_{\alpha \in \text{supp } \varphi(x)} |\alpha| |\varphi_{\ell,\alpha}| \right) \left(\max_{1 \leq s \leq \mu} \sum_{1 \leq j \leq n} \sum_{\vec{\beta} \in \coprod_{i=1}^n \text{supp } \frac{\partial \varphi_{\ell}(x)}{\partial x_i} \setminus \coprod_{s=1}^{\mu} \text{supp } M_s(x)} \max_{x \in \mathbf{K}} |h_{j,\ell}^{(s)}(x, a) x^{\vec{\beta}}| \right),$$

where $A \setminus B = \{\alpha - \beta \in \mathbb{Z}_{\geq 0}^n; \alpha \in A, \beta \in B\}$. The Relation (14) explains the summand of the above inequality. Therefore, if we set $C(a)$ as in Definition 3.1, we obtain the inequality (13). Evidently, $C(a)$ depends not on the coefficients of $\varphi_1(x), \dots, \varphi_n(x)$ but on the powers k and k' . This proves the invertibility of the matrix (11). Thus,

$$\begin{pmatrix} M_{\alpha_1}(x) \\ M_{\alpha_2}(x) \\ \vdots \\ M_{\alpha_{\mu}}(x) \end{pmatrix} = (id_{\mu} + \tau A(x))^{-1} \left(\sum_{i=1}^n H^{(i)} \frac{\partial}{\partial x_i} \right) \begin{pmatrix} f_1 + \tau \varphi_1 \\ f_2 + \tau \varphi_2 \\ \vdots \\ f_n + \tau \varphi_n \end{pmatrix}. \quad (15)$$

On the other hand:

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \vdots \\ \varphi_n(x) \end{pmatrix} = F \cdot \begin{pmatrix} \varphi(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = F \cdot G \begin{pmatrix} M_{\alpha_1}(x) \\ M_{\alpha_2}(x) \\ \vdots \\ M_{\alpha_{\mu}}(x) \end{pmatrix}$$

for some rank-1 ($n \times \mu$) polynomial matrix:

$$G = \begin{pmatrix} g_1(x) & g_2(x) & \cdots & g_{\mu}(x) \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

If we apply $F \cdot G$ from the left to the relation (15), we get:

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \vdots \\ \varphi_n(x) \end{pmatrix} = F \cdot G (id_{\mu} + \tau A(x))^{-1} \left(\sum_{i=1}^n H^{(i)} \frac{\partial}{\partial x_i} \right) \begin{pmatrix} f_1 + \tau \varphi_1 \\ f_2 + \tau \varphi_2 \\ \vdots \\ f_n + \tau \varphi_n \end{pmatrix}, \quad (16)$$

or

$$\begin{pmatrix} \varphi(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = G (id_{\mu} + \tau A(x))^{-1} \left(\sum_{i=1}^n H^{(i)} \frac{\partial}{\partial x_i} \right) \begin{pmatrix} f_1 + \tau \varphi_1 \\ f_2 + \tau \varphi_2 \\ \vdots \\ f_n + \tau \varphi_n \end{pmatrix}. \quad (17)$$

This relation gives rise to an inequality:

$$\begin{pmatrix} \varphi(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \sum_{i=1}^n \begin{pmatrix} 0 & \cdots & 0 & v_i^{\ell}(x, \tau) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \frac{\partial}{\partial x_i} \begin{pmatrix} f_1 + \tau \varphi_1 \\ f_2 + \tau \varphi_2 \\ \vdots \\ f_n + \tau \varphi_n \end{pmatrix}, \quad (18)$$

where the matrix in front of the derivative has single non zero ℓ -th column. That is to say we obtain the following equalities:

$$\begin{aligned}\varphi(x) &= \left(\sum_{i=1}^n v_i(x, \tau) \frac{\partial}{\partial x_i} \right) (f_\ell + \tau \varphi_\ell), \\ \varphi_\ell(x) &= F_{\ell 1} \varphi(x)\end{aligned}\tag{19}$$

The estimate (7) ensures that $v_i(x, \tau)$ are real analytic in $x \in \mathbf{K}$. Thus we have constructed a vector field corresponding to the homotopy we need. **Q.E.D.**

Example 3.1 *Let us consider the following system [4]:*

$$\begin{cases} f_1(x_1, x_2) = x_1^2 - x_2^2 - 1 = 0, \\ f_2(x_1, x_2) = x_1^2 + x_2^2 - 2 = 0. \end{cases}\tag{20}$$

This system has four real solutions within the square $[-2, 2]^2$:

$$(\pm\sqrt{1.5}, \pm\sqrt{0.5}) \approx (\pm 1.22474487139159, \pm 0.70710678118655).$$

If we perturb this system with a cubic monomial $\varphi(x) = x_1 x_2^2$ as follows:

$$\begin{cases} F_1(x_1, x_2) = x_1^2 - x_2^2 + t x_1 x_2^2 - 1 = 0, \\ F_2(x_1, x_2) = x_1^2 + x_2^2 - 2 = 0, \end{cases}\tag{21}$$

we calculate the constants $C(a) = \frac{5}{2}$, $\|\varphi\| = 3$, $\mu = 2$. Therefore we have four solutions of the system (21) if $t < \frac{1}{2 \cdot 3 \cdot 5}$. In particular, if we use the value $t = 0.033 < 1/30$ by applying the rootfinding method of [13] we obtain the following four solutions:

$$\begin{aligned} & (1.22054232589618, \pm 0.71433635683474), \\ & (-1.22879457180552, \pm 0.70004564158438). \end{aligned}$$

Furthermore, if we perturb both equations of the system (20) with the same cubic monomial $\varphi(x) = x_1 x_2^2$ as follows:

$$\begin{cases} F_1(x_1, x_2) = x_1^2 - x_2^2 + t x_1 x_2^2 - 1 = 0, \\ F_2(x_1, x_2) = x_1^2 + x_2^2 + t x_1 x_2^2 - 2 = 0, \end{cases}\tag{22}$$

we again calculate the constants $C(a) = \frac{5}{2}$, $\|\varphi\| = 3$, $\mu = 2$. Therefore we have four solutions of the system (22) if $t < \frac{1}{2 \cdot 3 \cdot 5}$. In particular, if we use the value $t = 0.033 < 1/30$ by applying the rootfinding method of [13] we obtain the following four solutions:

$$\begin{aligned} & (1.21652265747566, \pm 0.70710678118655), \\ & (-1.23302265747566, \pm 0.70710678118655). \end{aligned}$$

Example 3.2 *Let us consider the following system:*

$$\begin{cases} f_1(x_1, x_2, x_3) = \left(x_1^2 + \frac{x_2^2}{4} - x_3^2 \right) \left(\frac{x_1^2}{4} + x_2^2 - x_3^2 \right) - \frac{x_3^4}{81} = 0, \\ f_2(x_1, x_2, x_3) = (x_1 + x_2)^2 + 36(x_1 - x_2)^2 - 9x_3^2 = 0, \\ f_3(x_1, x_2, x_3) = \frac{x_1^2}{4} + x_2^2 + \frac{x_3^2}{9} - 1 = 0. \end{cases}\tag{23}$$

By applying the rootfinding method of [13] we obtain the following sixteen real solutions within the cube $[-2, 2]^3$:

$$\begin{aligned} & (0.62830967308983, 0.91412675198426, \pm 0.76883755100759), \\ & (-0.62830967308983, -0.91412675198426, \pm 0.76883755100759), \\ & (0.49635596537865, 0.91441703848857, \pm 0.95929271740718), \\ & (-0.49635596537865, -0.91441703848857, \pm 0.95929271740718), \\ & (1.11731818404380, 0.76796989195429, \pm 0.93973420474984), \\ & (-1.11731818404380, -0.76796989195429, \pm 0.93973420474984), \\ & (1.22450432822695, 0.66467487192937, \pm 1.28459776563576), \\ & (-1.22450432822695, -0.66467487192937, \pm 1.28459776563576). \end{aligned}$$

We observe that these roots are invariant under the actions of a group $G := (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ generated by two generators $(x_1, x_2, x_3) \mapsto (-x_1, -x_2, x_3)$ and $(x_1, x_2, x_3) \mapsto (x_1, x_2, -x_3)$ due to the invariance of the system (23) itself under the same group action. If we perturb this system with a quadratic monomial $\varphi(x) = x_2^2$ as follows:

$$\begin{cases} f_1(x_1, x_2, x_3) = \left(x_1^2 + \frac{x_2^2}{4} - x_3^2\right) \left(\frac{x_1^2}{4} + x_2^2 - x_3^2\right) - \frac{x_3^4}{81} = 0, \\ f_2(x_1, x_2, x_3) = (x_1 + x_2)^2 + 36(x_1 - x_2)^2 - 9x_3^2 + tx_2^2 = 0, \\ f_3(x_1, x_2, x_3) = \frac{x_1^2}{4} + x_2^2 + \frac{x_3^2}{9} - 1 = 0, \end{cases} \quad (24)$$

we calculate the constants $C(a) = \frac{1}{2}$, $\|\varphi\| = 2$, $\mu = 3$. Therefore we have sixteen solutions of the system (24) if $t < \frac{1}{3^2}$. In particular, if we use the value $t = 0.1 < 1/9$ by applying the rootfinding method of [13] we obtain the following sixteen real solutions within the cube $[-2, 2]^3$:

$$\begin{aligned} & (0.63087661393950, 0.91351892559324, \pm 0.77060795720733), \\ & (-0.63087661393950, -0.91351892559324, \pm 0.77060795720733), \\ & (0.49896002229193, 0.91405620623649, \pm 0.95934810309529), \\ & (-0.49896002229193, -0.91405620623649, \pm 0.95934810309529), \\ & (1.11568183127565, 0.76857206607484, \pm 0.93967783245553), \\ & (-1.11568183127565, -0.76857206607484, \pm 0.93967783245553), \\ & (1.22357424633595, 0.66527556809517, \pm 1.28379297777855), \\ & (-1.22357424633595, -0.66527556809517, \pm 1.28379297777855). \end{aligned}$$

One remarks here also the invariance of the roots under the above mentioned group G due to the invariance of the system (24) itself. \square

As for the equation (1) we establish the following theorem:

Theorem 3.2 *Let us consider a system of algebraic equations obtained as a perturbation of (1):*

$$\begin{cases} F_1(x_1, x_2, \dots, x_n) = 0, \\ F_2(x_1, x_2, \dots, x_n) = 0, \\ \vdots \\ F_n(x_1, x_2, \dots, x_n) = 0. \end{cases} \quad (25)$$

Suppose that on a ball $\mathbf{B}_r = \{x \in \mathbb{R}^n; |x| \leq r\}$ we have:

$$\text{rank} \left(\frac{\partial}{\partial x_j} f_k(x) \right)_{1 \leq j, k \leq n} = n. \quad (26)$$

Furthermore, we impose a condition on (F_1, F_2, \dots, F_n) :

$$\max_{x, y \in \mathbf{B}_r} \left\| \text{vector component of } \left(\frac{\partial}{\partial y_j} f_k(y) \right)_{1 \leq j, k \leq n}^{-1} \begin{bmatrix} (F_1 - f_1)(x) \\ (F_2 - f_2)(x) \\ \vdots \\ (F_n - f_n)(x) \end{bmatrix} \right\| < \varepsilon, \quad (27)$$

$$\max_{x, y \in \mathbf{B}_r} \left\| \text{vector component of } \left(\frac{\partial}{\partial y_j} F_k(y) \right)_{1 \leq j, k \leq n}^{-1} \begin{bmatrix} (F_1 - f_1)(x) \\ (F_2 - f_2)(x) \\ \vdots \\ (F_n - f_n)(x) \end{bmatrix} \right\| < \varepsilon, \quad (28)$$

where ε is strictly less than distance of any root of (1) in \mathbf{B}_r to the boundary $\partial \mathbf{B}_r$. Suppose that the system (1) has no multiple real roots.

Under these assumption the equality:

$$\#\{\text{real simple roots of (1) in } \mathbf{B}_r\} = \#\{\text{real simple roots of (25) in } \mathbf{B}_r\}$$

holds.

Proof We solve the homotopy equation with respect to smooth diffeomorphism

$$x_i + h_i(x, \tau), \quad 0 \leq i \leq n,$$

that satisfies

$$h_i(x, 0) = 0, \quad (29)$$

$$f_k(x_0 + h_0(x, \tau), \dots, x_n + h_n(x, \tau)) = f_k(x_0, \dots, x_n) + \tau(F_k(x) - f_k(x)), \quad 0 \leq \tau \leq 1. \quad (30)$$

The system gives rise to a system of $(n + 1)$ nonlinear differential equations:

$$\sum_{j=0}^n \frac{\partial h_j}{\partial \tau} \partial_j f_k(x_0 + h_0(x, \tau), \dots, x_n + h_n(x, \tau)) = F_k(x) - f_k(x), \quad (31)$$

for $k = 0, 1, \dots, n$. From the assumption (26), Eq. (31) is always solvable in the class of real analytic functions so far as

$$\det \left(\left(\frac{\partial}{\partial x_j} f_k(x) \right)_{1 \leq j, k \leq n} \right) \neq 0$$

after Cauchy–Kovalevskaya’s theorem [5] on the quasi-linear partial differential equation. After the conditions (27)-(28) and Eq. (31), $|\frac{\partial h_j}{\partial \tau}|$ is always strictly less than ε . Therefore $|h_j(\tau)| < \varepsilon \tau$ and $|x + h_j(\tau)| < |x| + \varepsilon \tau < r$ for x root of (1) located in the ball \mathbf{B}_r . Thus the homotopy equation admits a real analytic solution that connects $x \in \mathbf{B}_r$ with $x + h(x, 1) \in \mathbf{B}_r$. **Q.E.D.**

Corollary 3.1 *If the conditions on the analyticity of the homotopy constructed in the above Theorems 3.1 and 3.2 are fulfilled then none of the roots of the deformed system (35) crashes with another and consequently no new multiple roots are created after the proposed perturbation.*

Proof Assume that after the proposed perturbation a multiple root is created. Then the homotopy constructed in the above Theorems 3.1 and 3.2 loses its analyticity with respect to the parameter τ . **Q.E.D.**

4 Decomposition of multiple roots

In this section we recall facts about the decomposition of multiple roots into simple roots.

Definition 4.1 For the system (1), we define the Jacobian function:

$$(jf)(x) = \det \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}. \quad (32)$$

Let us denote by $V_{jf} = \{x \in \mathbb{R}^n : jf(x) = 0\}$ the zero set of $jf(x)$. We use the notation Q_{jf} for the following set

$$V_{jf} \cap \{x : (f_1^2 + \cdots + f_n^2)(x) = 0\}.$$

Then we have the following result:

Theorem 4.1 If $Q_{jf} = \emptyset$ then the system (1) has only simple roots, while if $Q_{jf} \neq \emptyset$ then the system (1) has multiple roots.

Proof The proof follows from well known facts in the singularity theory [2]. We prove the contrapositive of the statements. Namely we can easily see that the existence of multiple roots yields non-emptiness of Q_{jf} . On the other hand, the existence of only simple roots entails the emptiness of it Q_{jf} . **Q.E.D.**

Definition 4.2 Let us denote by (Jf) a vector valued ideal

$$(Jf) = \left\langle \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \mathbb{R}[x] \\ \vdots \\ \mathbb{R}[x] \end{bmatrix} \right\rangle. \quad (33)$$

Theorem 4.2 Let us consider a system like (1) for which we know that it possesses m real roots with multiplicities n_1, n_2, \dots, n_m where $1 \leq n_j$ for $1 \leq j \leq m$. Then there exists a vector polynomial:

$$\begin{bmatrix} H_1(x) \\ \vdots \\ H_n(x) \end{bmatrix} \in (\mathbb{R}[x])^n \Big/ (Jf) \quad (34)$$

such that the system of equations

$$\begin{aligned} (f_1 + H_1)(x) &= 0, \\ &\vdots \\ (f_n + H_n)(x) &= 0, \end{aligned} \quad (35)$$

has $n_1 + n_2 + \cdots + n_m$ simple real roots.

Proof We remark that

$$(f_i + H_i)(x) = \sum_{k=0}^{m_i} h_{i,k}(x') x_1^{m_i-k},$$

with $h_{i,m_i}(x') \neq 0, h_{i,0}(x') \neq 0, x' = (x_2, \dots, x_n)$, after certain permutation of variables x . It is well known that, there exists a perturbation $H_i(x)$ such that the equation $\sum_{k=0}^{m_i} h_{i,k}(x') x_1^{m_i-k} = 0$ has m_i simple roots for a codimension 1 set of x' (cf. [2]). This fact entails that the system (35) also possesses as much simple roots as (1) has. **Q.E.D.**

Remark 4.1 One can understand this theorem by an intuitive way. Let us denote by $I_i := \text{supp}(f_i + H_i)$ the set of powers present in the polynomial $f_i + H_i$. If the discriminant of the system

$$(f_i + H_i)(x) = \sum_{\alpha \in I_i} f_\alpha x^\alpha, \quad 1 \leq i \leq n, \quad (36)$$

say, $\Delta(f_\alpha) \in \mathbb{R}[f_{\alpha_1}, \dots, f_{\alpha_{\sum_{i=1}^n |I_i|}}]$ does not vanish, then the roots of the system (36) are all simple. That is to say the set of the coefficients of the system (36) for which the system has multiple roots is of codimension one in the space of coefficients $\mathbb{R}^{\sum_{i=1}^n |I_i|}$. This fact is known under the name of Bertini-Sard theorem [2].

Remark 4.2 It is worthy to notice that one shall choose a proper vector polynomial (34), to get distinct simple roots for the deformed system (35). For example if $f_1 = x^3$ the $f_1 = 0$ has a triple root at $x = 0$. If we take $H_1 = -x$ then $f_1 + H_1 = x^3 - x = x(x^2 - 1) = 0$ has 3 distinct roots $x = -1, 0, 1$, while for $H_1 = x$, the equation $x^3 + x = 0$ has 1 real simple root at $x = 0$ and two distinct complex roots.

Example 4.1 Let us consider the following system:

$$\begin{cases} f_1(x_1, x_2) = x_1^2 - x_2^2 - 1 = 0, \\ f_2(x_1, x_2) = x_1^4 + x_2^2 - 1 = 0. \end{cases} \quad (37)$$

This system has two multiple real solutions $(\pm 1, 0)$ within the square $[-2, 2]^2$.

If we perturb this system with $H_1 = 0$ and the simple linear polynomial $H_2 = t(x_1 - 2)$ where $0 < t \leq 0.5$ as follows:

$$\begin{cases} (f_1 + H_1)(x_1, x_2) = x_1^2 - x_2^2 - 1 = 0, \\ (f_2 + H_2)(x_1, x_2) = x_1^4 + x_2^2 + t(x_1 - 2) - 1 = 0, \end{cases} \quad (38)$$

then we have four simple real solutions. In particular, if we use the value $t = 0.5$ by applying the rootfinding method of [13] we obtain the following four real simple solutions:

$$\begin{aligned} & (1.07123233675477, \pm 0.38410769233261), \\ & (-1.20970135357686, \pm 0.68071827127359). \end{aligned}$$

While if we use the value $t = 0.025$ we obtain the following four real simple solutions:

$$\begin{aligned} & (1.00412951827050, \pm 0.09097301502177), \\ & (-1.01237171332486, \pm 0.15778620326351). \end{aligned}$$

Finally, if we use the value $t = 0.0125$ we obtain the following four real simple solutions:

$$\begin{aligned} & (1.00207398824224, \pm 0.06443817123186), \\ & (-1.00621769007449, \pm 0.11168724107454). \end{aligned}$$

Example 4.2 Let us consider the following system:

$$\begin{cases} f_1(x_1, x_2, x_3) = ((x_1 - x_2)^3 - x_3^2(x_1 + x_2 - x_3)) \\ \quad ((-x_1 - x_2)^3 - x_3^2(-x_1 + x_2 - x_3)) \\ \quad ((-x_1 + x_2)^3 - x_3^2(-x_1 - x_2 - x_3)) \\ \quad ((x_1 + x_2)^3 - x_3^2(x_1 - x_2 - x_3)) = 0, \\ f_2(x_1, x_2, x_3) = x_1^2 + x_2^2 - \frac{x_3^2}{2} = 0, \\ f_3(x_1, x_2, x_3) = x_1^2 + \frac{x_2^2}{9} + \frac{x_3^2}{4} - 1 = 0. \end{cases} \quad (39)$$

This system has eight simple real solutions:

$$(\pm 0.25926718242254, \pm 1.21300057180546, \pm 1.75418919109753),$$

and eight triple real solutions:

$$(\pm 0.68824720161168, \pm 0.68824720161168, \pm 1.37649440322337).$$

within the cube $[-2, 2]^3$. We observe that these roots are invariant under the actions of a group $\Gamma := (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ generated by three generators $(x_1, x_2, x_3) \mapsto (-x_1, x_2, x_3)$, $(x_1, x_2, x_3) \mapsto (x_1, -x_2, x_3)$ and $(x_1, x_2, x_3) \mapsto (x_1, x_2, -x_3)$ due to the invariance of the system (39) itself under the same group action.

If we perturb this system with H_1 where $0 < t \leq 0.5$ and $H_2 = H_3 = 0$ as follows:

$$\left\{ \begin{array}{l} (f_1 + H_1)(x_1, x_2, x_3) = \\ \quad ((x_1 - x_2)^3 - t(x_1 - x_2) - x_3^2(x_1 + x_2 - x_3)) \\ \quad ((-x_1 - x_2)^3 - t(-x_1 - x_2) - x_3^2(-x_1 + x_2 - x_3)) \\ \quad ((-x_1 + x_2)^3 - t(-x_1 + x_2) - x_3^2(-x_1 - x_2 - x_3)) \\ \quad ((x_1 + x_2)^3 - t(x_1 + x_2) - x_3^2(x_1 - x_2 - x_3)) = 0, \\ (f_2 + H_2)(x_1, x_2, x_3) = x_1^2 + x_2^2 - \frac{x_3^2}{2} = 0, \\ (f_3 + H_3)(x_1, x_2, x_3) = x_1^2 + \frac{x_2^2}{9} + \frac{x_3^2}{4} - 1 = 0, \end{array} \right. \quad (40)$$

then we have thirty two simple real solutions. In particular, if we use the value $t = 0.5$ we obtain the following eight real solutions which are shifts of the simple solutions to System (39):

$$(\pm 0.27142016486929, \pm 1.20645760731621, \pm 1.74883324771051).$$

Also we obtain the following twenty four simple real solution:

$$(\pm 0.68824720161168, \pm 0.68824720161168, \pm 1.37649440322337),$$

$$(\pm 0.78897550317143, \pm 0.32932116069209, \pm 1.20907797224513),$$

$$(\pm 0.44474589932680, \pm 1.07278013064881, \pm 1.64234961179579),$$

that originate from the triple solutions to System (39). We remark that the first ones of the above solutions coincide with the triple solutions to the original system. These roots are also invariant under actions of the group Γ . \square

5 Concluding remarks

A problem concerning the shift of roots of a system of algebraic equations has been investigated. Its conservation and decomposition of a multiple root into simple roots have been discussed.

To this end, with our central Theorem 3.1 we show that the number of real roots of a system located in a compact set does not change after a sufficiently small perturbation of the system. This theorem can be applied to high dimensional CAD where it is sometimes needed to calculate intersection of several hypersurfaces that are perturbation of a set of original (unperturbed) hypersurfaces. For example, to draw a 3D (three dimensional) picture of a real algebraic surface that obtained as a deformation of a known one, the question of the shift of roots plays essential role [7, §9.6]. We hope that our results to this direction represent certain interests to those who are interested in the application of algebraic equations to computer graphics.

Furthermore, we give a result about the decomposition of multiple roots into simple roots. In particular, our Theorem 4.2 assures us the existence of a deformed system (35) of the original system (1) that possesses only simple roots. This result can be used in many cases including the computation of the topological degree [4, 6, 10, 11, 12] in order to examine the solution set of a system of equations and to obtain information on the existence of solutions, their number and their nature [1, 3, 6, 8, 9].

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